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The problem of flame propagation in a random velocity field: weak turbulence limit

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Abstract. The problem of thin flame front propagation with curvature-dependent speed in a weak turbulent flow has been considered, and its connection with classical problems in the physics of disordered systems such as polymers in a random medium, growing interfaces and n-body interaction problems has been discussed. By a path-integral approach, an explicit formula for the randomly moving flame front in terms of the fluctuating velocity field has been derived. The steepest-descent approximation has been used to find the random configuration of the flame surface in the limit of small Markstein diffusivity. New expressions for the turbulent burning velocity (the overal propagation rate of a flame surface subject to a random velocity field) involving the random velocity field along the Lagrangian trajectories have been derived.

1. Introduction

In the last few years a flamelet formulation for premixed turbulent combustion has attracted considerable attention [1–4]. Under the flamelet assumption, one may describe the combustion process in the limiting case in which an infinitely fast chemical reaction leads to formation of thin wrinkled flame fronts embedded in the turbulent flow. Of particular prominence has been the approach which is based on a nonlinear stochastic differential equation (the G-equation) for the scalar $G(t, x)$ whose level surface moves normal to itself with a laminar burning velocity u_L [3, 5]

$$\frac{\partial G}{\partial t} + v(t, x) \cdot \nabla G = u_L |\nabla G| \tag{1}$$

where $v(t, x)$ is the random velocity field. Different techniques, such as numerical methods [5], the renormalization-group approach [6, 7], scaling and dimensional analysis [8, 9], and the spectral closure approximation [10] have been used to study the properties of the G-equation.

The main purpose of this paper is to derive an explicit expression for the flame surface position in terms of a weak random velocity field and examine the statistical properties of the flame front. A specific intention of this work is to derive the formula for the turbulent burning velocity and thereby to provide the framework for a study of the parametric dependence of the turbulent flame on the statistical characteristics of the prescribed random velocity field. There are two technical aspects to this paper. The first deals with the derivation of a linear differential equation for the field which is alternative to $G(t, x)$, while

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the second is concerned with the solution of the equation derived in terms of a path integral and its asymptotic analysis.

To the author's knowledge this work is a first attempt to formulate the problem of flame propagation in a random velocity field in terms of a path integral. Although it is impossible to evaluate such an integral analytically, a number of results can be obtained by considering a steepest-descent asymptotic and a replica method. The potential advantage of path integrals is that there exist effective numerical methods to evaluate them directly.

2. Regularization procedure

If the laminar burning velocity u_L is considered to be a constant, then the Huygens development of an initially smooth flame front develops singularities corresponding to the infinite curvature. However, if the G-equation (1) governing the propagation of the flame surface is modified in a manner that corresponds to the fact that any flame front in a real turbulent flow field would not be a pure discontinuity, but would have finite thickness, then the occurence of the cusps can be avoided. Such a regularization procedure may be applied considering the effect of flame stretch involving the flame thickness as a characteristic length scale. It is worth mentioning that as a physical phenomenon this situation is formally similar to that which occurs in a theory for the propagation of high-frequency waves through a random inhomogenious medium where focusing phenomena implying large fluctuations in wavefront curvature are likely to happen. In this situation, a regularization is usually achieved by the parabolic equation approximation [11].

Now let us consider the case in which the flame front has a finite thickness and therefore the laminar burning velocity is a function of strain and curvature [12, 13]. Under the conditions of weak strain and weak curvature, the formula for u_L can be written as follows

$$u_L = u_L^0 \left(1 + \frac{D_L}{u_L^0} \frac{\nabla^2 G}{|\nabla G|}\right) \tag{2}$$

where the Markstein diffusivity D_L may be regarded as a regularization parameter and u_L^0 is the laminar burning velocity for an unstretched flame surface. For further discussion and references on this aspect of the flame propagation see the article by Peters [10].

In this paper we restrict ourselves to a relatively simple case of the weak turbulence limit when the RMS velocity fluctuations, v_{RMS}, is smaller than the laminar burning velocity, u_L^0. Then for the combustion wave propagating in the z-direction, one can set [5–7]

$$G(t, \boldsymbol{x}) = -z + u_L^0 t + \varphi(t, \boldsymbol{r}) \qquad \varphi(0, \boldsymbol{r}) = \varphi_0(\boldsymbol{r}) \tag{3}$$

where $\boldsymbol{r} = (x, y)$ specifies the transverse coordinates. Thus the configuration of a fluctuating position of the flame front is described, at any instant t, by the single-valued function $\varphi(t, \boldsymbol{r})$ of the transverse vector $\boldsymbol{r}$. The single-valuedness assumption employed here is approximately valid as long as $v_{RMS} \ll u_L^0$.

Substitution of (3) in equations (1), (2), after expansion of the square root, yields

$$\frac{\partial \varphi}{\partial t} + \boldsymbol{v}_r(t, \boldsymbol{r}) \cdot \nabla_r \varphi = \frac{u_L^0}{2} (\nabla_r \varphi)^2 + D_L \nabla_r^2 \varphi + v_z(t, \boldsymbol{r}) \qquad \varphi(0, \boldsymbol{r}) = \varphi_0(\boldsymbol{r}) \tag{4}$$

where $\nabla_r = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$, $\boldsymbol{v}_r = (v_x, v_y)$. If $v_{RMS} \ll u_L^0$, the random velocity field can be considered as frozen. Therefore, in the frame of reference moving with a laminar burning velocity u_L^0 the components of the random fluid velocity $\boldsymbol{v}$ may be considered as the functions of the transverse vector $\boldsymbol{r}$ and time t. It is clear that the equation (4) is valid if the magnitude of gradient $|\nabla_r \varphi|$ is small. Note that the irrelevance of strain may be easily verified by considering the general formula for the laminar burning velocity of a flame submitted to stretch [10, 12, 13] in the limit of the small values of $|\nabla_r \varphi|$ and D_L.

3. Path-integral solution

An essential feature of the nonlinear equation (4) is that it may be converted into a linear equation. It is possible by using the transformation [14, 15]

$$\varphi = \frac{2D_L}{u_L^0} \ln \psi \tag{5}$$

which leads to a linear advection–diffusion equation for $\psi(t, \boldsymbol{r})$

$$\frac{\partial \psi}{\partial t} + \nabla_r \cdot \boldsymbol{v}_r(t, \boldsymbol{r})\, \psi) = D_L \nabla_r^2 \psi + \frac{u_L^0}{2D_L} v_z(t, \boldsymbol{r})\, \psi \qquad \psi(0, \boldsymbol{r}) = \psi_0(\boldsymbol{r})\,. \tag{6}$$

Here we have used the condition of incompressibility $\nabla_r \cdot \boldsymbol{v}_r = 0$.

To make further progress, let us introduce the stochastic differential equation

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s} = \boldsymbol{v}_r(s, \boldsymbol{r}(s)) + \boldsymbol{\xi}(s) \qquad 0 < s < t \tag{7}$$

with the boundary conditions

$$\boldsymbol{r}(0) = \boldsymbol{r}_0 \qquad \boldsymbol{r}(t) = \boldsymbol{r} \tag{8}$$

where $\boldsymbol{\xi}(s)$ is a white Gaussian noise vector with zero mean and with a probability density functional of the form [16]

$$P_\xi = \exp\left(-\frac{1}{4D_L} \int \xi^2(s)\, \mathrm{d}s\right). \tag{9}$$

Then the solution of the initial value problem (6) may be represented as a function space integral obtained by averaging over random realizations of $\boldsymbol{r}(s)$ [16, 17]

$$\psi(t, \boldsymbol{r}) = \int \delta(\boldsymbol{r}(t) - \boldsymbol{r}) \exp\left[\frac{u_L^0}{2D_L} \int_0^t v_z(s, \boldsymbol{r}(s))\, \mathrm{d}s\right] P[\boldsymbol{r}(s)]\, \psi_0(\boldsymbol{r}_0)\, \mathcal{D}\boldsymbol{r}(s) \tag{10}$$

where the integration is performed over all trajectories $\boldsymbol{r}(s)$ starting at $s = 0$ with $\boldsymbol{r}(0) = \boldsymbol{r}_0$ and arriving at the point $\boldsymbol{r}$ at time t. It should be noted that the solution (10) is valid for any fixed realization of the random velocity field. The weighting factor $P[\boldsymbol{r}(s)]$ may be readily found from (7), (9)

$$P[\boldsymbol{r}(s)] = J \exp\left(-\frac{1}{4D_L} \int_0^t \left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s} - \boldsymbol{v}_r(s, \boldsymbol{r}(s))\right)^2 \mathrm{d}s\right).$$

Here $J = \exp(\alpha \int_0^t \nabla_r \cdot \boldsymbol{v}_r\ \mathrm{d}s)$ is the Jacobian corresponding to mapping $\boldsymbol{r}(s)$ onto $\boldsymbol{\xi}(s)$ [16, 17] which is equal to unity for an incompressible fluid.

It is convenient to use the action functional

$$S[\boldsymbol{v}(t, \boldsymbol{r}(t)), \boldsymbol{r}(t)] = \int_0^t L[\boldsymbol{v}(s, \boldsymbol{r}(s)), \boldsymbol{r}(s)]\, \mathrm{d}s + S_0 \tag{11}$$

with the Lagrangian L of the form

$$L = \frac{1}{4}\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s} - \boldsymbol{v}_r(s, \boldsymbol{r}(s))\right)^2 - \frac{u_L^0}{2} v_z(s, \boldsymbol{r}(s)). \tag{12}$$

and $S_0 = -(u_L^0/2)\, \varphi_0(\boldsymbol{r}_0)$.

Since the flame position is described by $\varphi = (2D_L/u_L^0)\ln\psi$, the final result may be written as

$$\varphi(t, r) = \frac{2D_L}{u_L^0}\ln\int\delta(r(t) - r)\exp\left(-\frac{S[v(s, r(s)), r(s)]}{D_L}\right)\mathcal{D}r(s)\,. \qquad (13)$$

The major advantage of this formula is that it gives us an explicit expression for the fluctuating flame front $\varphi(t, r)$ in terms of the random velocity field $v(t, r)$. It should be noted that we have introduced no assumption concerning the statistical properties of the random velocity field, hence the formula (13) is quite general and can be evaluated further if we use explicit statistics of the velocity $v(t, r)$. Before considering this, a specific analogy might be helpful in illustrating the ideas and formulas presented so far. If the fluctuating velocity field in the transverse direction v_r is zero (this assumption is commonly used in combustion literature), then the auxiliary field $\psi(t, r)$ may be interpreted as the restricted partition function of a Gaussian polymer of length t; the fluctuating position of the flame front $\varphi(t, r)$ plays the role of a polymer free energy, while the velocity field in the z-direction, v_z, has the formal meaning of the random potential. Recently, such a similiarity in the context of growing interfaces, directed polymers, etc, has been used successfully [15]. In the absence of transverse velocity fluctuations one can take an advantage of the fact that equation (4) is completely equivalent to the Kardar–Parisi–Zhang (KPZ) model [14].

4. Replica method

Now let us sketch the algorithm that can be used to investigate the influence of non-zero velocity fluctuations transverse to the reaction zone. To illustrate the idea, consider the average value of φ over random velocity field

$$\langle\varphi\rangle = \frac{2D_L}{u_L^0}\langle\ln\psi\rangle\,.$$

To perform the average of the logarithm of the 'partition function' ψ is a classical problem in the physics of disordered systems [18]. One way to solve it is a replica method based on the identity [19–21]

$$\langle\ln\psi\rangle = \lim_{n\to 0}\frac{\langle\psi^n\rangle - 1}{n} \qquad (14)$$

where $\psi^n \equiv \prod_{i=1}^n \psi(t, r_i)$ may be written as a functional integral over n paths $r_i(s)$

$$\psi^n = \int\exp\left\{-\frac{1}{4D_L}\int_0^t\left[\sum_{i=1}^n\left(\frac{\mathrm{d}r_i}{\mathrm{d}s} - v_r(s, r_i(s))\right)^2 - \frac{u_L^0}{2}v_z(s, r_i(s))\right]\mathrm{d}s\right\}$$
$$\times\prod_{i=1}^n\delta(r_i(t) - r_i)\,\mathcal{D}r_i(s)\,. \qquad (15)$$

There is, however, a difficulty associated with the fact that for non-zero field v_r (which appears in quadratic form in (15)) the path integral (13) is not the same as the expression for 'ordinary' free energy. This problem may be solved through introduction of n auxilliary vectors $p_i = (p_{xi}, p_{yi})$ [22, 23] which make the path integral (15) 'linear' in v_r

$$\psi^n = \int\int\exp\left\{\int_0^t\left[\sum_{i=1}^n -p_i\cdot\frac{\mathrm{d}r_i}{\mathrm{d}s} + D_L p_i^2 + p_i\cdot v_r(s, r_i(s)) + \frac{u_L^0}{8D_L}v_z(s, r_i(s))\right]\mathrm{d}s\right\}$$
$$\times\prod_{i=1}^n\delta(r_i(t) - r_i)\,\mathcal{D}r_i(s)\,\mathcal{D}(ip_i(s))\,. \qquad (16)$$

This expression greatly simplifies obtaining an average value of ψ^n. Let us assume that the velocity v is a Gaussian random field of the form [24]

$$v = (0,\ v_y(s,x),\ v_z(s,x)) \tag{17}$$

with zero mean and correlation functions given by

$$\begin{aligned}\langle v_y(s,x)\, v_y(s',x')\rangle &= 2Y(x-x')\delta(s-s')\\ \langle v_z(s,x)\, v_z(s',x')\rangle &= 2Z(x-x')\delta(s-s')\,.\end{aligned} \tag{18}$$

Then by using the well known formula for zero mean Gaussian variables $\langle\exp\xi\rangle = \exp(\langle\xi^2\rangle/2)$ we obtain

$$\langle\psi^n\rangle = \int\int \exp\left\{\int_0^t \left[\sum_{i=1}^n -p_i\cdot\frac{\mathrm{d}r_i}{\mathrm{d}s} + \sum_{i,j=1}^n D_{\mathrm{L}}\delta_{ij}p_{xi}p_{xj} + (D_{\mathrm{L}}\delta_{ij} + Y(x_i-x_j))p_{yi}p_{yj} + \left(\frac{u_{\mathrm{L}}^0}{8D_{\mathrm{L}}}\right)^2 Z(x_i-x_j)\right]\mathrm{d}s\right\}\prod_{i=1}^n \delta(r_i(t)-r_i)\,\mathcal{D}r_i(s)\,\mathcal{D}(ip_i(s)) \tag{19}$$

where δ_{ij} is the Kronecker delta. Note that this procedure can be extended in a straightforward manner to the case in which the statistical properties of the velocity field are determined by a two-point correlation tensor.

The formula (19) is nothing but a path-integral solution of an 'n-body' interaction problem

$$\frac{\partial\langle\psi^n\rangle}{\partial t} = H_n\langle\psi^n\rangle \tag{20}$$

where the 'n'-body Hamiltonian H_n is

$$H_n = \sum_{i,j=1}^n \left[D_{\mathrm{L}}\delta_{ij}\frac{\partial^2}{\partial x_i\partial x_j} + (D_{\mathrm{L}}\delta_{ij} + Y(x_i-x_j))\frac{\partial^2}{\partial y_i\partial y_j} + \left(\frac{u_{\mathrm{L}}^0}{8D_{\mathrm{L}}}\right)^2 Z(x_i-x_j)\right].$$

Since the asymptotic behaviour of $\langle\psi^n\rangle$ is determined by the largest eigenvalue of (20) E_n, the long-time properties of a random flame surface should relate to the properties of $\lim_{n\to 0} E_n$.

For

$$Y(x_i-x_j) = Y\delta_{ij} \tag{21}$$

equation (20) has been extensively studied in particular in the context of the replica method [20, 21]. It is clear that this case corresponds to a simple renormalization of the Markstein diffusivity and hence the model considered here will have the same long-distance behaviour as the KPZ model [14].

5. Small Markstein diffusivity limit

It is clear from (20) and (21) that the influence of non-zero velocity fluctuations transverse to the reaction zone might be the same as that of diffusion term. It is reasonable therefore to consider the case of small Markstein diffusivity D_{L}. In this case $\varphi(t,r)$ can be expressed by the minimum of the action S. It follows from (13) that in the limit $D_{\mathrm{L}}\to 0$, the dominant contribution to the space function integral comes from those realizations of the random process $r(s)$ which minimize the action S [17, 25–27]. Therefore the functional integral may be estimated by the steepest-descent method which gives an expression for $\varphi_*(t,\boldsymbol{r}) = \lim_{D_{\rm L}\to 0}\varphi(t,\boldsymbol{r})$

$$\varphi_*(t,\boldsymbol{r}) = -\frac{2}{u_{\rm L}^0}\min S[\boldsymbol{v}(s,\boldsymbol{r}(s)),\boldsymbol{r}(s)] \tag{22}$$

where a minimum has to be taken over all trajectories satisfying the boundary conditions (8).

If we assume the initial condition for $\varphi(t,\boldsymbol{r})$ to be zero, then according to (11) we may rewrite (22) as

$$\varphi_*(t,\boldsymbol{r}) = -\frac{2}{u_{\rm L}^0}\int_0^t L[\boldsymbol{v}(s,\boldsymbol{r}_*(s)),\boldsymbol{r}_*(s)]\,{\rm d}s \tag{23}$$

where $\boldsymbol{r}_* = \boldsymbol{r}_*(s)$ is a solution of the Euler–Lagrange equation associated with the action (11)

$$\frac{\rm d}{{\rm d}s}\frac{\partial L}{\partial\dot{\boldsymbol{r}}} = \frac{\partial L}{\partial\boldsymbol{r}} \qquad 0 < s < t \qquad \left(\dot{\boldsymbol{r}} \equiv \frac{{\rm d}\boldsymbol{r}}{{\rm d}s}\right) \tag{24}$$

with the conditions (8) .

It is also clear that the flame configuration $\varphi_*(t,\boldsymbol{r})$ may be found as a solution of a nonlinear first-order differential equation of Hamilton–Jacobi type

$$\frac{\partial\varphi_*}{\partial t} + H[\varphi_*,\nabla_r\varphi_*] = 0 \tag{25}$$

where H is the Legendre transformation of the Lagrangian (12) multiplied by $-2/u_{\rm L}$

$$H = \boldsymbol{v}_r(t,\boldsymbol{r})\cdot\nabla_r\varphi_* - \frac{u_{\rm L}^0}{2}(\nabla_r\varphi_*)^2 - v_z(t,\boldsymbol{r})\,. \tag{26}$$

What we have seen here is that the constant laminar burning velocity assumption has a very natural interpretation in terms of the path integral. In the semiclassical approximation, when

$$\psi \sim \exp\left(-\frac{\min S}{D_{\rm L}}\right) \tag{27}$$

the curvature has no effect on the flame front position φ.

6. Turbulent burning velocity

Now we turn to the problem of the turbulent burning velocity $u_{\rm T}$ which is the overall propagation rate of a flame in a random flow. Since the flame front is assumed to be thin, the enhancement of the burning rate by a random velocity field must be entirely owing to the increase in the flame surface. Therefore $u_{\rm T}$ can be expressed in terms of the measure of the interface roughness $\langle(\nabla_r\varphi)^2\rangle$ [5]

$$u_{\rm T} = u_{\rm L}^0\left(1 + \tfrac{1}{2}\langle(\nabla_r\varphi)^2\rangle\right) \tag{28}$$

where angular brackets denote the ensemble averaging. From (13), it follows that the gradient $\nabla_r\varphi$ may be written as

$$\nabla_r\varphi = \frac{2}{u_{\rm L}^0}\frac{\int\nabla_r S{\rm e}^{-\frac{S}{D_{\rm L}}}\,\mathcal{D}\boldsymbol{r}(s)}{\int{\rm e}^{-\frac{S}{D_{\rm L}}}\,\mathcal{D}\boldsymbol{r}(s)}\,. \tag{29}$$

Note that this formula has similar mathematical structure to the Hopf–Cole solution of the Burgers equation [28]. Substitution of (29) into (28) gives an expression for $u_{\rm T}$ which, however, would be too general to work with. But in the limit $D_L \rightarrow 0$, an approximate solution of (29) is $\nabla_r \varphi \approx (2/u_L^0)\nabla_r S(r_*(s))$ and, therefore,

$$u_T = u_L^0 + \frac{2}{u_L^0} \left\langle \left(\int_0^t \nabla_r L \, ds \right)^2 \right\rangle \tag{30}$$

where of course L is a function of s along $r_*(s)$. This expression may be considered as the generalization of the well known Clavin–Williams formula [29]. By using (24), the formula for the turbulent burning velocity u_T may be rewritten in terms of 'generalized momenta' $\partial L/\partial \dot{r}$ as

$$u_T = u_L^0 + \frac{2}{u_L^0} \left\langle \left(\frac{\partial L}{\partial \dot{r}} \right)^2 \right\rangle . \tag{31}$$

In particular, when $v_r = 0$, it follows from (12), (24), (31) that

$$u_T = u_L^0 + \frac{2}{u_L^0} \left\langle \left(\frac{dr}{ds}(t) \right)^2 \right\rangle \tag{32}$$

where $r = r(s, t, r)$ is a solution of the Newton equation

$$\frac{d^2 r}{ds^2} = -u_T^0 \nabla_r v_z(s, r(s)) \qquad 0 < s < t \tag{33}$$

with the boundary conditions (8).

To find an average value of the 'kinetic energy' along the Lagrangian trajectory $r(s)$ in (32) and hence u_T, it is necessery to specify a random velocity field v_z that plays the role of a potential field. It would be interesting to use (30) to derive the $\frac{4}{3}$-power dependence of u_T on the ratio of the RMS velocity fluctuation, u_{RMS}, to the laminar flame velocity, u_L^0, previously deduced by simple dimensional analysis [9] (see also [30]).

7. Summary

We have shown that the problem of flame propagation with curvature-dependent speed in a weak random flow field can be formulated in terms of a path integral. Even though we cannot evaluate such an integral analytically, a number of results have been obtained by considering the limit of small Markstein diffusivity when the path integral can be estimated by the steepest-descent method. Also, the replica method has been applied to perfom the average of the logarithm of the path integral. It should be noted that powerful numerical methods exist to evaluate the functional integrals directly [31–33]. Therefore, it might appear that the path integral (13) may be more useful numerically than the original formulation based on a stochastic partial differential equation (4). A description of turbulent flames in terms of a path integral leads quite directly to an analogy between turbulent flame problems and those that occur in the physics of disordered media. Besides technical advantages, such a similarity may lead to a deeper understanding of a turbulent flamelet combustion. In particular, the results obtained in this paper are of direct relevance to the experiments on the spatial fluctuations of the flame contour in an SI-engine combustion chamber [34], since they provide a strategy for obtaining the universal characteristics of scale-invariant fluctuations of a flame surface. We have also derived the new expressions for the turbulent burning velocity, which is the overall propagation rate of a flame surface subject to random velocity field.

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